Common Final Exam

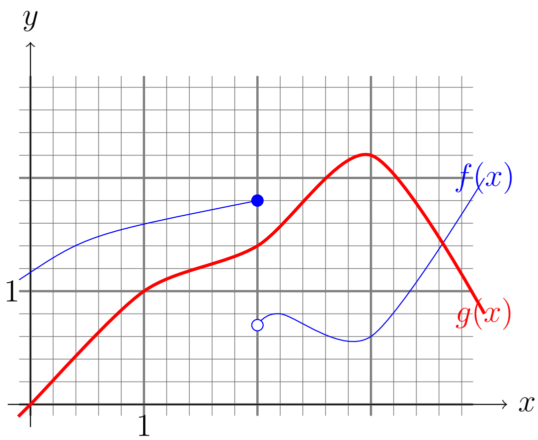
Calculus I, Math 161, Fall 2023

Name: SOLUTIONS

Question	Points	Score
1	18	
2	10	
3	10	
4	20	
5	10	
6	10	
7	10	
8	20	
9	10	
10	10	
11	20	
12	10	
13	20	
Total:	178	

- No books or notes of any kind are allowed
- No technology – calculators, cell phones, or computers – is allowed
- Show your work!
- You have 120 minutes to complete this exam.

1. The functions $f(x)$ and $g(x)$ have the following graphs:



Based on these graphs, compute the following limits if they exist.

(a) (3 points) $\lim\limits_{x \to 2^+} f(x)$ 0.7

(b) (3 points) $\lim\limits_{x \to 2^-} g(x)f(x)$ $= 1.4 \times 1.8 \ (= 2.52)$

(c) (3 points) $\lim\limits_{x \to 2} g(x)f(x)$ DNE

(d) (3 points) $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)}$ DNE

(e) (3 points) $\lim\limits_{x \to 0} \dfrac{g(x)}{f(x)}$ $= \dfrac{0}{1.2} = 0$

(f) (3 points) $\lim\limits_{x \to 3} f(g(x))$ $= \lim\limits_{x \to 2.2} f(x) = 0.8$

2. (10 points) Find a function $f(x)$ and a point a so that the derivative of $f(x)$ at a is given by

$$\lim_{h\to 0} \frac{\sin((e+h)^2) - \sin(e^2)}{h}$$

$f(x) = \sin(x^2)$

& $a = e$

has the property that

$f'(e) = \lim\limits_{h\to 0} \dfrac{\sin((e+h)^2) - \sin(e^2)}{h}$

3. (10 points) Determine the values of constants a and b that make the following function differentiable everywhere

$$f(x) = \begin{cases} a\sin(\pi x) & x \le -1 \\ \sqrt{3x+7} + b & x > -1 \end{cases}$$

$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} a\sin(\pi x) = a\cdot\sin(-\pi) = 0$

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} \sqrt{3x+7} + b = \sqrt{4} + b = 2 + b$

so $2 + b = 0$ ie $b = -2$

$\lim\limits_{x\to -1^-} f'(x) = \lim\limits_{x\to -1^-} \pi\cdot a\cos(\pi x) = \pi\cdot a\cdot\cos(-\pi) = -a\pi$

$\lim\limits_{x\to -1^+} f'(x) = \lim\limits_{x\to -1^+} \dfrac{3}{2\sqrt{3x+7}} = \dfrac{3}{2\sqrt{4}} = \dfrac{3}{4}$

so $-a\pi = \dfrac{3}{4}$ ie $a = \dfrac{-3}{4\pi}$

4. Evaluate derivatives of the following functions with respect to x:

(a) (5 points) $y = x^5 + x^3 + x + x^{-1} + x^{-3} + x^{-5}$

$y' = 5x^4 + 3x^2 + 1 - x^{-2} - 3x^{-4} - 5x^{-6}$

(b) (5 points) A, B, C are constants:

$$f(x) = (Ax^3 + B\sqrt{x} + C)^4$$

chain rule: $f'(x) = 4(Ax^3 + B\sqrt{x} + C)^3 \cdot \left(3Ax^2 + \frac{B}{2\sqrt{x}}\right)$

(c) (5 points) $g(x) = e^{-3x}\sin(x^2)$

product rule & chain rule

$g'(x) = -3e^{-3x} \cdot \sin(x^2) + e^{-3x} \cdot \cos(x^2) \cdot 2x$

(d) (5 points) $y = \ln\left(x\sqrt{x^2+1}\right)$

chain rule + product rule + chain rule:

$$\frac{\sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}}}{x\sqrt{x^2+1}}$$

OR

use log rules:

$y = \ln(x) + \frac{1}{2}\ln(x^2+1)$

$y' = \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot (2x)$

5. (10 points) Find the absolute max and min value, and the points where they occur, for the function $g(x) = e^{x^3-3x^2-1}$ on the interval $[0, 3]$.

$g'(x) = (3x^2 - 6x) \cdot e^{x^3-3x^2-1}$

$g'(x) = 0 \Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x-2) = 0$

$\Rightarrow x = 0$ or $x = 2$

potential max & mins:

$x =$	$g(x) =$
0	e^{-1}
2	$e^{8-12-1} = e^{-5}$ ← min
3	$e^{27-27-1} = e^{-1}$

MAX

6. (10 points) Find the equation of the tangent line to the following curve at $(2,3)$. Please put your answer in slope intercept form.

   $x^3 + y^2 + xy = 23$

$3x^2 + 2yy' + 1\cdot y + x\cdot y' = 0 \Rightarrow$

$(2y+x)y' = -y - 3x^2 \Rightarrow$

$y' = \dfrac{-y-3x^2}{2y+x}$

at $(2,3)$, $y' = \dfrac{-3-3\cdot 4}{2\cdot 3+2} = \dfrac{-15}{8}$

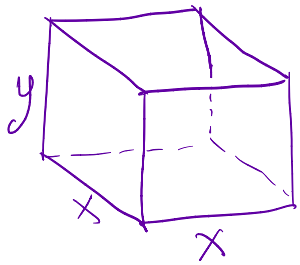
line thru $(2,3)$ w/ slope $= \frac{-15}{8}$ is

$y - 3 = -\frac{15}{8}(x-2)$,

ie

$y = -\frac{15}{8}x + \frac{27}{4}$

7. (10 points) We are creating a cardboard box without a top which has a square base. If we have 150 square feet of cardboard to use, find the dimensions of the box which maximize its volume. Please include the proper units.



Vol $= x^2 y$ ft^3

surface area $= x^2 + 4xy = 150$ ft^2

Solve S.A. for y: $y = \dfrac{150 - x^2}{4x}$

Then $Vol(x) = x^2 \cdot \dfrac{150 - x^2}{4x} = \dfrac{150}{4}x - \dfrac{1}{4}x^3$

Set $Vol'(x) = 0$:

$$\frac{150}{4} - \frac{3}{4}x^2 = 0 \implies$$

$$150 = 3x^2 \implies$$

$$x = \sqrt{50}$$

$x = \sqrt{50}$ feet,

$y = \dfrac{150 - 50}{4\sqrt{50}} = \dfrac{100}{4\sqrt{50}} = \dfrac{100\sqrt{50}}{200} = \dfrac{\sqrt{50}}{2}$ feet.

8. Evaluate the following limits

(a) (5 points) $\lim_{x\to 0} \frac{\tan(2x)}{\tan(3x)}$ $\left(= \frac{0}{0} : \text{use L'Hôpital's Rule}\right)$

$$= \lim_{x\to 0} \frac{2\cdot \sec^2(2x)}{3\cdot \sec^2(3x)} = \frac{2\cdot \sec^2(0)}{3\cdot \sec^2(0)} = \frac{2}{3}$$

(b) (5 points) $\lim_{x\to\infty} \frac{\sqrt{x^4+1}}{x^2}$ $= \lim_{x\to\infty} \sqrt{\frac{x^4+1}{x^4}}$

$$= \sqrt{\lim_{x\to\infty} \frac{x^4+1}{x^4}} = \sqrt{1} = 1$$

(c) (5 points) $\lim_{x\to 5} \frac{\sqrt{4+x}-\sqrt{4}}{x}$ $= \frac{\sqrt{9}-\sqrt{4}}{5} = \frac{3-2}{5} = \frac{1}{5}$

(d) (5 points) $\lim_{x\to 1} \frac{(x^2-1)(x^2-4)(x^2-9)}{(x-1)(x-2)(x-3)}$ $= \lim_{x\to 1} \frac{(x-1)(x+1)(x^2-4)(x^2-9)}{(x-1)(x-2)(x-3)} =$

$$= \lim_{x\to 1} \frac{(x+1)(x^2-4)(x^2-9)}{(x-2)(x-3)} = \frac{2\cdot(-3)\cdot(-8)}{(-1)\cdot(-2)}$$

$$= 24$$

9. (10 points) Use an appropriately chosen linearization to estimate $\ln(0.95)$.

Linearize $y = \ln(x)$ at $x = 1$:

$y' = \frac{1}{x}$

$L(x)$ goes thru $(1, \ln(1)) = (1, 0)$

with slope $\frac{1}{1} = 1$: $L(x) = 0 + 1(x-1)$

$= x - 1$

So, $\ln(0.95) \approx L(0.95) = 0.95 - 1 = -0.05$

10. (10 points) Which function is an antiderivative of $f(x) = \dfrac{2x+2}{x^2+2x+5}$?

(a) $\dfrac{x^2+2x}{(1/3)x^3+x^2+5x}$

(b) $\dfrac{2}{2x+2}$

(c) $\ln(x^2+2x+5)$

Because $\frac{d}{dx}(\ln(x^2+2x+5)) = \frac{1}{x^2+2x+5} \cdot (2x+2)$

$= f(x)$

11. Evaluate the following integrals. Please fully simplify your answer.

(a) (10 points) $\int x^{-1/3} - x^{1/3} dx$ $=$

$$\frac{3}{2} \cdot x^{2/3} - \frac{3}{4} \cdot x^{4/3} + C$$

(b) (10 points) $\int_0^2 x^2 e^{x^3} dx$

$u = x^3 \qquad du = 3x^2 dx$

$\frac{1}{3} du = x^2 dx$

$$D = \int_{x=0}^{2} e^u \cdot \frac{1}{3} du = \frac{1}{3} e^u \Big|_{x=0}^{2} = e^{x^3} \Big|_0^2 =$$

$$= \frac{1}{3}(e^8 - e^0) = \frac{e^8 - 1}{3}$$

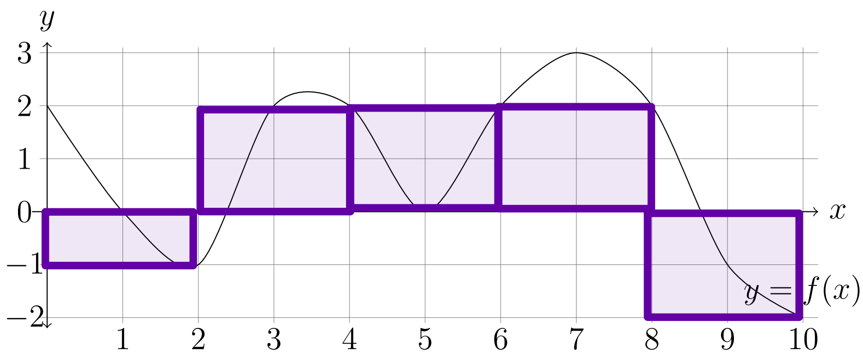
12. (10 points) Water is being added to a swimming pool at a rate of $r(t) = 6t^2 + 6t$ gallons per hour, where t is the time in hours after midnight. If there were already 30 gallons of water in the pool at midnight, how many gallons of water are in the pool at 2am? Please use proper units.

Volume of water $= 30 + \int_0^2 6t^2 + 6t \, dt$

$= 30 + \left(2t^3 + 3t^2 \Big|_0^2\right)$

$= 30 + (16 + 12 - 0) = 58$ gallons

13. Below is a graph of $f(x)$.



(a) (5 points) Calculate the right Riemann sum with $n = 5$ rectangles for $\int_0^{10} f(x)\,dx$.

$$\int_0^{10} f(x)dx \approx 2\cdot f(2) + 2\cdot f(4) + 2\cdot f(6) + 2\cdot f(8) + 2\cdot f(10)$$
$$= -2 + 4 + 4 + 4 - 4 = 6$$

For the next three parts, let $F(x) = \int_2^x f(x)\,dx$.

(b) (5 points) Estimate $F(8)$.

$F(8)$ = area under curve between $x = 2$ and $x = 8$:

$F(8) \approx 10$ by counting boxes, $F(8) \approx 12$ by Riemann sum w/ 3 right rectangles

(c) (5 points) Estimate $F(0)$.

$$F(0) = \int_2^0 f(x)dx = -\int_0^2 f(x)dx = -(0.5)$$

(d) (5 points) Estimate $F'(4)$.

$$F'(4) = f(4) = 2$$